[3] H. E. Green, "The numerical solution of some important transmission-line problems," *IEEE Trans. Microwave Theory Tech.* (*Special Issue on Microwave Filters*), vol. MTT-13, pp. 676–692, Sept. 1965.
[4] M. V. Schneider, "Computation of impedance and attenuation of TEM-lines by finite difference methods," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 793–800, Nov. 1965.
[5] J. W. Duncan, "The accuracy of finite-difference solutions of Laplace's equation," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-15, pp. 575–582, Oct. 1967.
[6] D. H. Sinnot, "The use of interpolation in improving finite difference solutions of TEM mode structures," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 20–28, Jan. 1969.
[7] D. H. Sinnot, G. K. Cambell, C. T. Carson, and H. E. Green, "The finite difference solutions of microwave circuit problems," *IEEE Trans. Microwave Theory Tech.* (*Special Issue on Computer-Oriented Microwave Practices*), vol. MTT-17, pp. 464–478, Aug. 1969.
[8] T. Hatsuda and T. Matsumoto, "Computation of impedance of partially filled and slotted coaxial line," *IEEE Trans. Microwave Theory Tech.* (Corresp.), vol. MTT-15, pp. 643–644, Nov. 1967.
[9] T. Hatsuda *et al.*, "Computation of characteristic impedance of inhomogeneous slotted coaxial lines by relation method," *Electron. Commun. Japan*, vol. 51-B, no. 4, pp. 91–99, Apr. 1968; also (in Japanese), *J. Inst. Elec. Eng.* (*Tokyo*), vol. 51-B, no. 4, pp. 156–163, Apr. 1968.
[10] D. L. Gish and D. Graham, "Characteristic impedance and phase velocity of a dielectric-supported air strip transmission line with side wall," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 131–148, Mar. 1970.

Solutions for Some Waveguide Discontinuities by the Method of Moments

VU KHAC THONG

***Abstract*—The electromagnetic boundary value problem of two waveguides coupled by an aperture or an aperture in a waveguide radiating into free space may be described by an integral equation. An analytical solution to this integral equation cannot be readily found due to the complexity of the kernel. However, extremely useful results may be obtained if the method of moments is employed to reduce the integral equation to a matrix equation which can be solved by known methods. In this short paper, series and shunt slots in a rectangular waveguide are analyzed using this technique.**

The electromagnetic boundary value problem of two waveguides coupled by an aperture or an aperture in a waveguide radiating into free space may be considered to be solved if the tangential components of the electric field at the aperture are determined for various excitation conditions. For a system of two waveguides coupled by an aperture S, the integral equation for the tangential components of the electric field at the aperture can be shown to be [1]

$$n \times H^{\rm inc} = j\omega\epsilon n \times \int_s [\overline{G}_h^{(1)}(r \mid r_0) + \overline{G}_h^{(2)}(r \mid r_0)] \cdot [n \times E(r_0)]\, ds_0. \quad (1)$$

$\overline{G}_h^{(1)}, \overline{G}_h^{(2)}$ are the magnetic Green's dyadics for the separate waveguides satisfying equations such as

$$\nabla \times \nabla \times \overline{G}_h^{(1)}(r \mid r_0) - k^2\overline{G}_h^{(1)}(r \mid r_0) = -\overline{I}\delta(r - r_0)$$
$$n \times \nabla \times \overline{G}_h^{(1)}(r \mid r_0) = 0$$

at the guide walls. $H^{\rm inc}$ is the magnetic field of the exciting mode. It is assumed that the waveguides are connected to matched loads.

Analytical solution to the integral equation (1) may not be readily found due to the complexity of the kernel. However, extremely useful approximate results may be obtained if it is noted that (1) has the form

$$Lf = (L^{(1)} + L^{(2)})f = g \quad (2)$$

where $L, L^{(1)}, L^{(2)}$ are linear operators. By the method of moments [2] (2) can be reduced to an Nth-order matrix equation. If the basis func-

Manuscript received June 14, 1971; revised November 29, 1971.
The author is with the Department of Electrical Engineering, Monash University, Clayton, Victoria 3168, Australia.

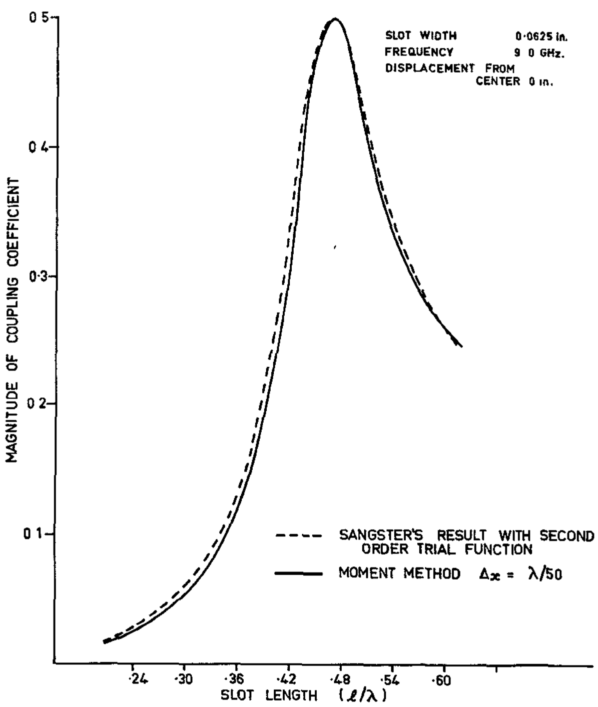

Fig. 1. Magnitude of coupling coefficient as a function of slot length for a series slot in the broadwall coupling two rectangular waveguides.

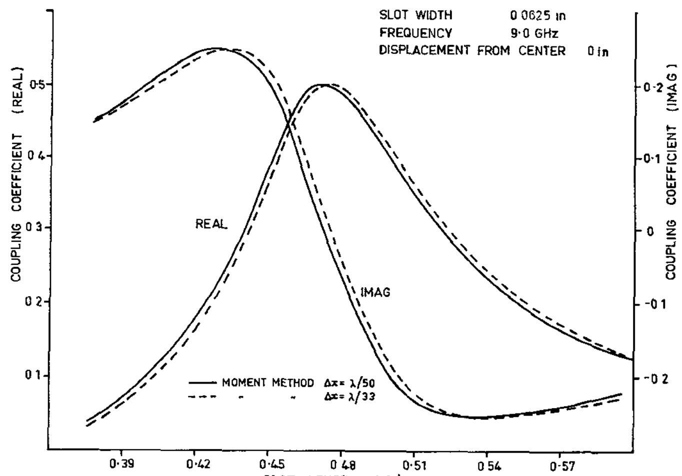

Fig. 2. Coupling coefficient as a function of slot length for a series slot in the broadwall coupling two rectangular waveguides.

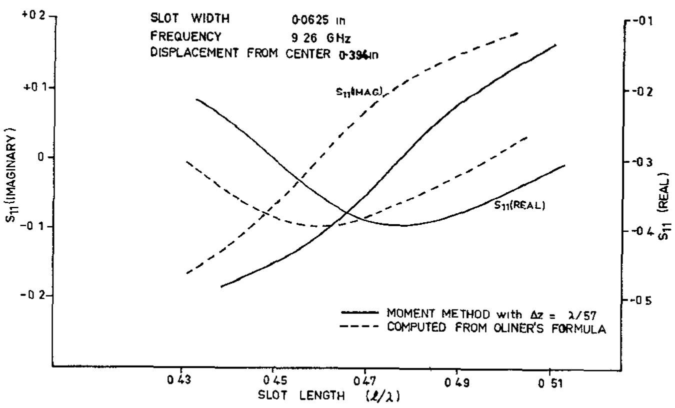

Fig. 3. Reflection coefficient of dominant mode due to a shunt slot in the broadwall of a rectangular waveguide.

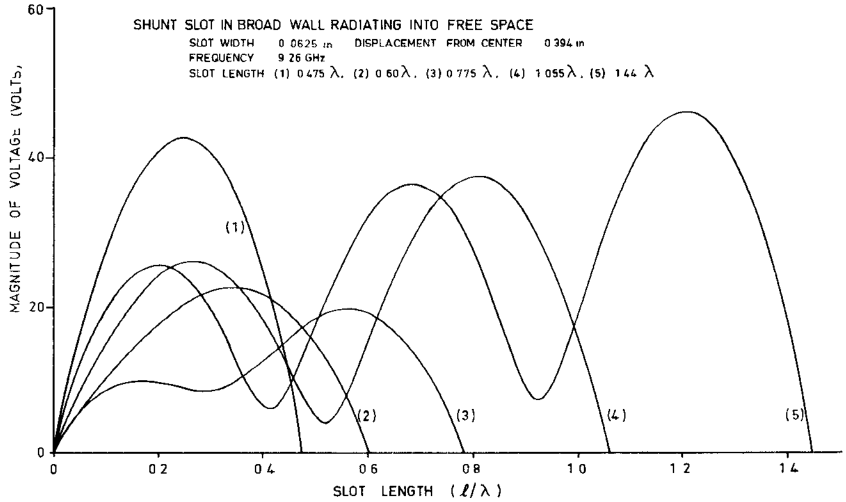

Fig. 4. Variation of the magnitude of voltage across a shunt slot with slot length.

tions are pulse functions defined by

$$f_i(x, y, z) = a_x \text{ (or } a_z), \qquad \text{for } x_i - \frac{\Delta x}{2} < x < x_i + \frac{\Delta x}{2}$$

$$z_i - \frac{\Delta z}{2} < z < z_i + \frac{\Delta z}{2}$$

$$= 0, \qquad \text{otherwise}$$

and point matching is used (this may be regarded as testing with Dirac delta functions w_i), then for an aperture in the broadwall of a rectangular waveguide a typical term of the matrix may be written explicitly as

$$z_{ij} = \langle w_i, Lf_j \rangle = \langle w_i, L^{(1)}f_j \rangle + \langle w_i, L^{(2)}f_j \rangle$$

$$\langle w_i, L^{(1)}f_j \rangle = \sum_{\substack{n=0 \\ n \neq m=0}}^{\infty} \sum_{m=0}^{\infty} \frac{\epsilon_{0mn}}{ab} \times \cos\left(\frac{n\pi x_i}{a}\right) \cos\left(\frac{m\pi y_i}{b}\right) \cos\left(\frac{m\pi y_j}{b}\right)$$

$$\times \frac{a}{n\pi} \times \left[\sin \frac{n\pi \left(x_j + \frac{\Delta x}{2}\right)}{a} - \sin \frac{n\pi \left(x_j - \frac{\Delta x}{2}\right)}{a} \right]$$

$$\times \left[-\frac{j\omega\epsilon}{\Gamma_{mn}^2} + \frac{1}{j\omega\mu} \right]$$

$$\times \left[\exp\left(-\Gamma_{mn} \left| z_i - \left(z_j + \frac{\Delta z}{2}\right) \right| \right) - \exp\left(-\Gamma_{mn} \left| z_i - \left(z_j - \frac{\Delta z}{2}\right) \right| \right) \right]$$

$$\epsilon_{0mn} = 1, \qquad \text{if } m \text{ or } n = 0$$

$$= 2, \qquad \text{otherwise}$$

$$y_i = y_j = b$$

$\langle w_i, L^{(2)}f_j \rangle$ has a similar form.

Numerical Results

Numerous computations have been performed for rectangular waveguide slot antennas and waveguide broadwall couplers using the above method. X-band waveguides (0.4 by 0.9 inch internal dimensions) with zero wall thickness are used throughout. It is also assumed that only one component of the electric field at the slots need be taken into account since the lots are narrow. Results for a centered series slot coupling two identical waveguides are shown in Figs. 1 and 2. Excellent agreement with Sangster's results [3] is obtained. It is interesting to note that Sangster's results, based on a variational formula, are almost exact in this case since the trial function used approximates the actual solution extremely well. Fig. 2 examines the effect of varying the number of pulse functions used for a given slot length. It can be seen that no significant improvement in results is obtained in reducing the pulse function length from $\Delta x = \lambda/33$ to $\Delta x = \lambda/50$. Fig. 3 compares the results obtained by this method and those obtained from Oliner's theory [4] for a shunt slot antenna in the broad face of a rectangular waveguide. It can be seen that close agreement would be obtained if the curves computed from Oliner's formula are shifted forward by $\lambda/60$. This discrepancy is not unexpected because no allowance is made for the variation of resonant length with displacement from the waveguide center in Oliner's formulation, although it is known that the resonant length is about 5 percent higher at the edge. It is found that at resonance, the electric field for every point along the shunt slot is in phase, confirming a well-known experimental result. A 20-percent change in slot length from resonant condition will result in a phase shift of 30 degrees between the edges of a shunt slot although the variation of the magnitude of the electric field is still essentially sinusoidal. A complete plot of the variation of the electric field (or more precisely $\int_{\text{slot width}} E \cdot dl$) along a shunt slot for slots of various lengths are shown in Figs. 4 and 5. Pulse functions with $\Delta z = \lambda/28$ are used. The variation of the coupling coefficient and scattering matrix element S_{11} with slot length is presented in Fig. 6. The plane passing through the center of the slot is used as reference throughout.

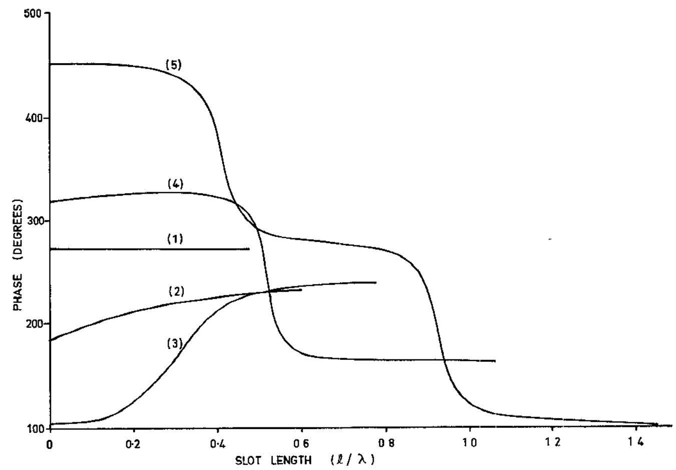

Fig. 5. Variation of the phase of voltage across a shunt slot with slot length. Same data as in Fig. 4.

The method discussed in this short paper is readily applicable to shunt and series slots of any length as well as to large apertures with complex shapes. It is also quite flexible since a change in aperture boundary requires only partial reevaluation of the coefficient matrix

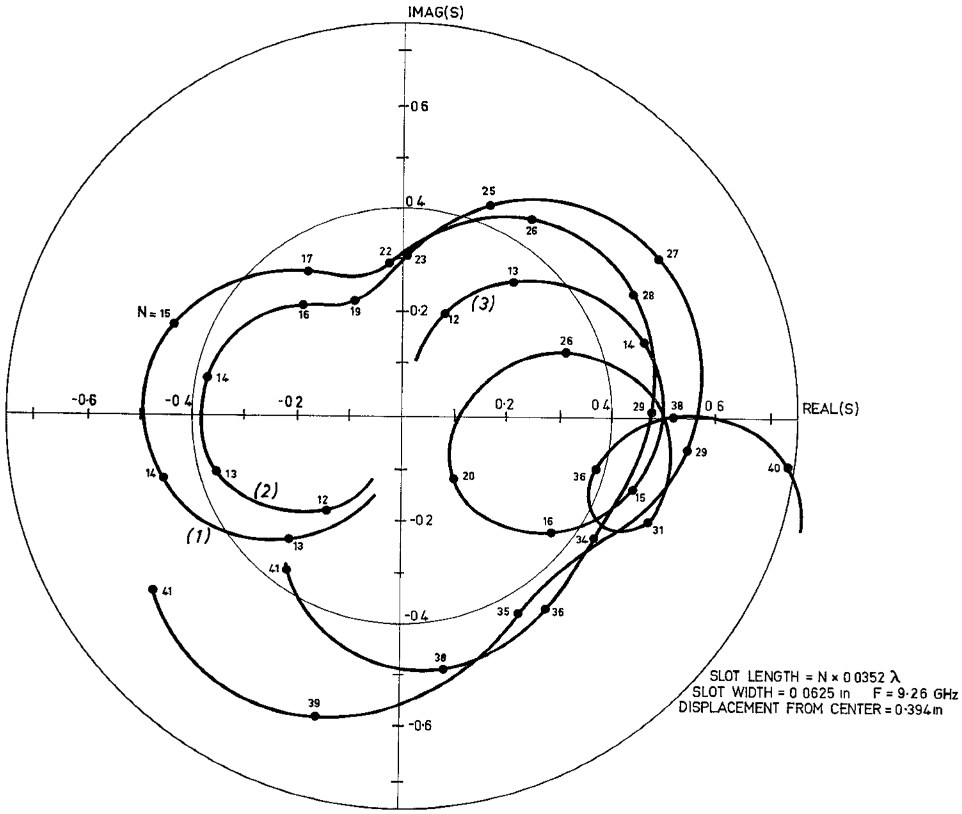

Fig. 6. Variation of coupling coefficient and reflection coefficient with slot length. (*1*), S_{11} of a shunt slot in broadwall coupling two identical waveguides; (*2*), S_{11} of a shunt slot in broadwall radiating into free space; (*3*), coupling coefficient of a shunt slot in broadwall coupling two identical waveguides.

and provides as its solution the electric field at the aperture, in addition to the various parameters of interest such as coupling coefficient, junction impedance, etc.

ACKNOWLEDGMENT

The author gratefully acknowledges discussions with G. K. Cambrell and C. T. Carson concerning this work.

REFERENCES

[1] R. E. Collin and F. J. Zucker, *Antenna Theory*, vol. 1. New York: McGraw-Hill, 1969, ch. 14.
[2] R. F. Harrington, *Field Computation by Moment Methods*. New York: Macmillan, 1968.
[3] A. J. Sangster, "Variational method for the analysis of waveguide coupling," *Proc. Inst. Elec. Eng.*, vol. 112, pp. 2171–2179, Dec. 1965.
[4] A. A. Oliner, "The impedance properties of narrow radiating slots in the broad face of rectangular waveguide, Part I—Theory; Part II—Comparison with measurement," *IRE Trans. Antennas Propagat.*, vol. AP-5, pp. 4–20, Jan. 1957.

A Generalized Locking Equation for Oscillators

JOHN P. QUINE

***Abstract*—Locking equations are derived which account for nonsinusoidal device waveforms. Locking bandwidth is related to Q values and device voltage amplitudes. Effective Q values are calculated for cavities having tuned and untuned harmonics.**

Manuscript received November 22, 1971; revised January 12, 1972.
The author is with the Research and Development Center, General Electric Company, Schenectady, N. Y.

Adler [1] has derived a relation between the reflection phase shift and the gain of locked oscillators based on the assumption of a sinusoidal device voltage waveform. However, to achieve high efficiency in present-day solid-state microwave power sources, it is required in many cases that the device current and voltage waveforms contain strong harmonic components. In these cases, Adler's equation does not apply. In this short paper, the derivation of a generalized locking equation is presented which accounts for the presence of strong harmonic components, and allows the prediction of the locking bandwidth for an arbitrary cavity configuration and for arbitrary device waveforms.

A single-valued static i–v characteristic is assumed. For this case, the area described by the instantaneous operating point as it moves along the i–v curve during one fundamental period is zero [2]. Thus

$$\int_0^T i\,dv = 0, \tag{1}$$

where $T = 1/f = 2\pi/\omega$ is the fundamental period of oscillation, and

$$v = \sum_{n=1}^{\infty} V_n \sin(2n\pi ft + \alpha_n) \tag{2}$$

$$i = \sum_{n=1}^{\infty} I_n \sin(2n\pi ft + \beta_n) \tag{3}$$

$$dv = \sum_{n=1}^{\infty} n\omega V_n \cos(2n\pi ft + \alpha_n)dt. \tag{4}$$

Performing the integration indicated by (1) yields Groszkowski's result [2]:

$$\sum_{n=1}^{\infty} n\,|V_n|\,|I_n|\,\sin(\alpha_n - \beta_n) = 0. \tag{5}$$

Equation (5) states that the total reactive power flow is zero, and can be expressed as